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# Free convection in a shallow cavity with variable properties—2. Porous media

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**Abstract**—Free convection in a porous shallow cavity with differentially heated end walls has been studied. The governing differential equations are analytically solved by applying the method of asymptotic expansions. The results show that the constant-property solution (Boussinesq approximation) deviates approximately 3% from the variable-property solution, if the properties in the Boussinesq solution are taken as the arithmetic mean between the hot and cold end wall temperature.

## 1. INTRODUCTION

IN ANALYTIC studies on free convection problems the fluid properties are usually treated as constants except the density in the buoyancy term of the equation of motion. This assumption, which is generally referred to as the Boussinesq approximation, is reasonable if the appearing temperature and pressure gradients are not too large.

In opposition to forced convection problems, the density has to be considered as temperature dependent in free convection problems, because temperature and density gradients, respectively, are the generator of the motion. For that reason it could be generally expected that the variable properties have a bigger effect on free than on forced convection.

A systematic study on the effect of variable properties on free and forced convection boundary layer problems was carried out by Herwig [1]. Merker and Mey [2] solved the governing equation for free convection in a shallow cavity with variable properties using the method of matched asymptotic expansions. Blythe and Simpkins [3] theoretically studied the free convection in a porous layer with a temperature-dependent viscosity. In the present study, free convection in a porous shallow cavity with differentially heated end walls is considered. This very simple geometry has already been examined in the studies carried out by Walker and Homsy [4] as well as by Bejan and Tien [5]. The governing equations are solved by applying the method of asymptotic expansions, as was already done in Part 1 of this paper for a Newtonian fluid.

## 2. MATHEMATICAL FORMULATION

Figure 1 shows a schematic sketch of the shallow cavity. The temperature and velocity field in the core region are also sketched in this figure. The end walls are kept at constant but different temperatures $T_c$ and $T_h$, respectively. The cavity is filled with a porous medium.

The governing equations for this free convection problem are the equation of continuity

$$A\frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=0 \tag{1}$$

the equation of motion

$$\eta u=-Gr\frac{\partial p}{\partial x} \tag{2}$$

$$A\eta v=-Gr\frac{\partial p}{\partial y}+(1-\rho)\frac{\rho_R g\kappa}{u_R\eta_R} \tag{3}$$

and the equation of thermal energy

$$Gr\,Pr\,\rho c_p\left[A^2u\frac{\partial\theta}{\partial x}+Av\frac{\partial\theta}{\partial y}\right]=A^2\frac{\partial}{\partial x}\left\{\lambda\frac{\partial\theta}{\partial x}\right\}+\frac{\partial}{\partial y}\left\{\lambda\frac{\partial\theta}{\partial y}\right\}. \tag{4}$$

These equations have to be solved subject to the boundary conditions

$$x=0:\quad u=0,\quad \theta=\theta_c$$
$$x=1:\quad u=0,\quad \theta=\theta_h$$
$$y=0,1:\quad v=0,\quad \frac{\partial\theta}{\partial y}=0.$$

Equations (1)–(4) have already been made dimensionless by using the quantities† given in Table 1. As dimensionless groups appear the Grashof number, $Gr=\rho_R u_R h/\eta_R$, the Prandtl number, $Pr=v_R/a_R$ and the cavity aspect ratio $A=h/l$.

The variable properties are developed in a Taylor series as was done elsewhere [1, 2]. One obtains finally, for example, for the density

†The asterisk refers to physical quantities.

**NOMENCLATURE**

| | |
|---|---|
| $a$ | thermal diffusivity [$m^2 s^{-1}$] |
| $c_p$ | heat capacity [$J kg^{-1} K^{-1}$] |
| $g$ | acceleration of gravity [$m s^{-2}$] |
| $h$ | height of cavity [m] |
| $l$ | length of cavity [m] |
| $p$ | pressure [$N m^{-2}$] |
| $T$ | temperature [K] |
| $u, v$ | velocity components [$m s^{-1}$] |
| $x, y$ | Cartesian coordinates [m]. |

Greek symbols

| | |
|---|---|
| $\beta$ | coefficient of thermal expansion [$K^{-1}$] |
| $\varepsilon$ | perturbation parameter [—] |
| $\eta$ | dynamic viscosity [Pa s] |
| $\theta$ | dimensionless temperature [—] |
| $\kappa$ | permeability [$m^2$] |
| $\lambda$ | thermal conductivity [$W m^{-1} K^{-1}$] |
| $\nu$ | kinematic viscosity [$m^2 s^{-1}$] |
| $\rho$ | density [$kg m^{-3}$] |
| $\psi$ | stream function [—]. |

Dimensionless groups

| | |
|---|---|
| $A$ | cavity aspect ratio |
| $Gr$ | Grashof number |
| $K_{\alpha_i}$ | dimensionless property of the second kind |
| $Nu$ | Nusselt number |
| $Pr$ | Prandtl number. |

Subscripts

| | |
|---|---|
| B | Boussinesq approximation |
| c | cold |
| h | hot |
| R | reference state. |

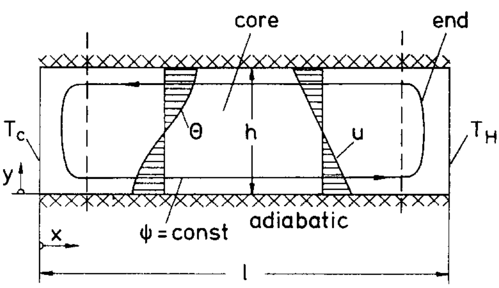


FIG. 1. Schematic sketch of the porous shallow cavity.

$$\rho = 1 + \varepsilon K_{\rho_1}\theta + \frac{\varepsilon^2}{2} K_{\rho_2}\theta^2 \tag{5}$$

with

$$K_{\rho_1} = \left(\frac{T}{\rho^*}\frac{\partial \rho^*}{\partial T}\right)_R \tag{6}$$

$$K_{\rho_2} = \left(\frac{T^2}{\rho^*}\frac{\partial^2 \rho^*}{\partial T^2}\right)_R \tag{7}$$

and

$$\varepsilon = \frac{T_h - T_c}{T_R}. \tag{8}$$

For the reference temperature we choose the arithmetic mean between the end wall temperatures, $T_R = (T_c + T_h)/2$, as well as the cold end temperature, $T_R = T_c$. The reference velocity $u_R$ is determined by considering the buoyancy term in equation (3). Substituting the Taylor series (5) for the properties in equations (1)–(4) and demanding that all terms in equation (3) are of $O(1)$ for small $\varepsilon$, one obtains

$$u_R = -\frac{\rho_R g \kappa \varepsilon K_{\rho_1}}{\eta_R}. \tag{9}$$

Considering the identity

$$\varepsilon K_{\rho_1} = -\beta_R (T_h - T_c) \tag{10}$$

as well as the expression for the kinematic viscosity $\nu_R = \eta_R/\rho_R$, one finds for the reference velocity

$$u_R = \frac{g\kappa\beta_R(T_h - T_c)}{\nu_R} \tag{11}$$

and, finally, for the Grashof number

$$Gr = \frac{g\kappa h}{\nu_R}\beta_R(T_h - T_c) \tag{12}$$

an expression, frequently used in porous media problems.

Table 1. Dimensionless quantities

| $x$ | $y$ | $u$ | $v$ | $p$ | $\theta$ | $\rho$ | $\eta$ | $\lambda$ | $c_p$ |
|---|---|---|---|---|---|---|---|---|---|
| $\frac{x^*}{l}$ | $\frac{y^*}{h}$ | $\frac{u^*}{Au_R}$ | $\frac{v^*}{Au_R}$ | $\frac{Kp^*}{\rho_R u_R^2 h^2}$ | $\frac{T - T_R}{T_H - T_R}$ | $\frac{\rho^*}{\rho_R}$ | $\frac{\eta^*}{\eta_R}$ | $\frac{\lambda^*}{\lambda_R}$ | $\frac{c_p^*}{c_{pR}}$ |

## 3. SOLUTION TECHNIQUE

The governing equations (1)–(4) are solved by using the method of asymptotic expansions which is explained in detail elsewhere and will not be repeated again here. We speak of linear theory if terms to $O(\varepsilon^1)$, and of quadratic theory if terms to $O(\varepsilon^2)$ are considered. Finally, one ends up at $O(\varepsilon^0)$ with the following set of equations:

$$A\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial y}=0 \tag{13}$$

$$u_0=-Gr\frac{\partial p_0}{\partial x} \tag{14}$$

$$Av_0=-Gr\frac{\partial p_0}{\partial y}+\theta_0 \tag{15}$$

$$Gr\,Pr\left(A^2u_0\frac{\partial\theta_0}{\partial x}+v_0\frac{\partial\theta_0}{\partial y}\right)=A^2\frac{\partial^2\theta_0}{\partial x^2}+\frac{\partial^2\theta_0}{\partial y^2}. \tag{16}$$

These equations have already been deduced by Walker and Homsy as well as by Bejan and Tien by applying the Boussinesq approximation to the governing equations.

After some manipulations one ends for the $O(\varepsilon^1)$ the *equation of continuity*

$$A\frac{\partial u_1}{\partial x}+\frac{\partial v_1}{\partial y}+AK_{\rho_1}\frac{\partial\theta_0}{\partial x}u_0+K_{\rho_1}\frac{\partial\theta_0}{\partial y}v_0=0 \tag{17}$$

the *Darcy equations*

$$K_{\eta_1}u_0\theta_0+u_1=-Gr\frac{\partial p_1}{\partial x} \tag{18}$$

$$A(K_{\eta_1}v_0\theta_0+v_1)=-Gr\frac{\partial p_1}{\partial y}+\theta_1+\frac{1}{2}\frac{K_{\rho_2}}{K_{\rho_1}}\theta_0^2 \tag{19}$$

and the *equation of thermal energy*

$$Gr\,Pr\left[A^2u_0\frac{\partial\theta_1}{\partial x}+A^2u_1\frac{\partial\theta_0}{\partial x}+A^2K_{\rho_1}\theta_0u_0\frac{\partial\theta_0}{\partial x}\right.$$
$$+A^2K_{c_{p1}}\theta_0u_0\frac{\partial\theta_0}{\partial x}+Av_0\frac{\partial\theta_1}{\partial y}+Av_1\frac{\partial\theta_0}{\partial y}$$
$$\left.+AK_{\rho_1}\theta_0v_0\frac{\partial\theta_0}{\partial y}+AK_{c_{p1}}\theta_0v_0\frac{\partial\theta_0}{\partial y}\right]$$
$$=A^2\frac{\partial^2\theta_1}{\partial x^2}+A^2K_{\lambda_1}\left(\frac{\partial\theta_0}{\partial x}\right)^2+A^2K_{\lambda_1}\theta_0\frac{\partial^2\theta_0}{\partial x^2}$$
$$+\frac{\partial^2\theta_1}{\partial y^2}+K_{\lambda_1}\left(\frac{\partial\theta_0}{\partial y}\right)^2+K_{\lambda_1}\theta_0\frac{\partial^2\theta_0}{\partial y^2}. \tag{20}$$

The set of equations (13)–(16) has already been solved by Bejan and Tien as well as by Walker and Homsy using the method of asymptotic expansions with the cavity aspect ratio $A$ as the expansion parameter. In the remainder it is shown how the resulting equations at $O(\varepsilon^1)$ and $O(\varepsilon^2)$ are solved using this method.

It is appropriate to introduce the stream function

$$\rho u=\frac{\partial\psi}{\partial y},\quad \rho v=-A\frac{\partial\psi}{\partial x} \tag{21}$$

which identically fulfils the equation of continuity. In addition, the pressure is eliminated in the usual way by applying the curl operation to the equations of motion. For further details see refs. [2, 4, 5].

## 4. SOLUTIONS

For the case $T_R=(T_h+T_c)/2$ one obtains for the core velocity component

$$u=\frac{1}{2}(2y-1)\left[1+\varepsilon\left(x-\frac{1}{2}\right)\left(\frac{K_{\rho_2}}{K_{\rho_1}}-K_{\eta_1}-K_{\lambda_1}\right)\right.$$
$$+\varepsilon^2\left\{\left(x-\frac{1}{2}\right)^2\left(K_{\eta_1}^2+K_{\lambda_1}K_{\eta_1}-\frac{K_{\rho_2}}{K_{\rho_1}}K_{\eta_1}\right.\right.$$
$$\left.+\frac{K_{\rho_2}}{K_{\rho_1}}K_{\lambda_1}\right)+\frac{x}{2}(x-1)\left(\frac{K_{\rho_2}}{K_{\rho_1}}K_{\lambda_1}-K_{\eta_1}K_{\lambda_1}\right)$$
$$\left.\left.+\frac{1}{2}\left(x^2-x+\frac{1}{6}\right)(3K_{\lambda_1}^2-K_{\lambda_2})\right\}\right]. \tag{22a}$$

The heat transport in the core region is due to diffusion and convection. Furthermore, as the lower and upper walls of the cavity are adiabatic, the heat flux in each cross-section of the cavity must remain the same. Hence, the Nusselt number can be calculated from

$$Nu=\int_0^1\left(\lambda\frac{\partial\theta}{\partial x}-Gr\,Pr\,\rho uc_p\theta\right)\mathrm{d}y. \tag{23}†$$

With $T_R=(T_c+T_h)/2$ one obtains for the Nusselt number

$$Nu=1+\varepsilon^2\frac{K_{\lambda_2}}{24}+\frac{A^2Gr^2Pr^2}{120}$$
$$\times\left[1+\varepsilon^2\left(-\frac{1}{2}K_{\rho_1}K_{\lambda_1}-\frac{1}{2}\frac{K_{\rho_2}}{K_{\rho_1}}K_{\lambda_1}+\frac{1}{2}K_{\eta_1}K_{\lambda_1}\right.\right.$$
$$+\frac{1}{2}K_{\lambda_1}^2-\frac{3}{4}K_{c_{p1}}K_{\lambda_1}+\frac{1}{4}K_{\eta_1}^2-\frac{1}{3}\frac{K_{\rho_2}}{K_{\rho_1}}K_{\eta_1}$$
$$-\frac{1}{3}K_{\rho_1}K_{\eta_1}-\frac{K_{\eta_2}}{12}+\frac{5}{12}K_{\rho_2}+\frac{1}{2}\frac{K_{\rho_2}}{K_{\rho_1}}K_{c_{p1}}$$
$$+\frac{1}{2}K_{\rho_1}K_{c_{p1}}-\frac{1}{2}K_{c_{p1}}K_{\eta_1}+\frac{K_{c_{p2}}}{6}-\frac{K_{\lambda_2}}{24}+\frac{1}{6}K_{c_{p1}}^2$$
$$\left.\left.+\frac{K_{\rho_1}^2}{12}+\frac{1}{12}\left(\frac{K_{\rho_2}}{K_{\rho_1}}\right)^2\right)\right]. \tag{24a}$$

† One should remember that the properties $\lambda$, $\rho$, and $c_p$ are dimensionless quantities, see Table 1.

In the case of $T_R = T_c$ one obtains for the core velocity component

$$u = \frac{1}{2}(2y-1)\left[1+\varepsilon\left[\frac{K_{\lambda_1}}{2}+x\left\{\frac{K_{\rho_2}}{K_{\rho_1}}-K_{\eta_1}-K_{\lambda_1}\right\}\right]\right.$$
$$+\varepsilon^2\left[\frac{K_{\lambda_2}}{6}+x\left\{\frac{K_{\rho_2}}{K_{\rho_1}}K_{\lambda_1}-K_{\eta_1}K_{\lambda_1}-K_{\lambda_1}^2\right\}\right.$$
$$+x^2\left\{\frac{3}{2}K_{\eta_1}K_{\lambda_1}-\frac{K_{\rho_2}}{K_{\rho_1}}K_{\eta_1}+K_{\eta_1}^2\right.$$
$$\left.\left.\left.-\frac{K_{\eta_2}}{2}+\frac{3}{2}K_{\lambda_1}^2-\frac{K_{\lambda_2}}{2}-\frac{3}{2}\frac{K_{\rho_2}}{K_{\rho_1}}K_{\lambda_1}\right\}\right]\right] \quad (22b)$$

and for the Nusselt number

$$Nu = 1+\varepsilon\frac{K_{\lambda_1}}{2}+\varepsilon^2\frac{K_{\lambda_2}}{6}+\frac{A^2\,Gr^2\,Pr^2}{120}$$
$$\times\left[1+\varepsilon\left(K_{\rho_1}+\frac{K_{\rho_2}}{K_{\rho_1}}-K_{\eta_1}-\frac{1}{2}K_{\lambda_1}+\frac{3}{2}K_{c_{p1}}\right)\right.$$
$$+\varepsilon^2\left(\frac{13}{4}K_{\lambda_1}^2-\frac{2}{3}K_{\lambda_2}-\frac{13}{6}K_{\rho_1}K_{\lambda_1}-\frac{13}{6}\frac{K_{\rho_2}}{K_{\rho_1}}K_{\lambda_1}\right.$$
$$+\frac{13}{6}K_{\eta_1}K_{\lambda_1}-\frac{5}{3}K_{\eta_1}K_{\lambda_1}-\frac{5}{3}\frac{K_{\rho_2}}{K_{\rho_1}}K_{\eta_1}+\frac{4}{3}K_{\eta_1}^2$$
$$-\frac{K_{\eta_2}}{2}+\frac{13}{6}K_{\rho_2}-\frac{5}{3}K_{c_{p1}}K_{\lambda_1}+\frac{7}{3}K_{\rho_1}K_{c_{p1}}$$
$$+\frac{7}{3}\frac{K_{\rho_2}}{K_{\rho_1}}K_{c_{p1}}-\frac{7}{3}K_{\eta_1}K_{c_{p1}}+\frac{5}{6}K_{c_{p2}}+\frac{K_{\rho_1}^2}{3}$$
$$\left.\left.+\frac{1}{3}\left(\frac{K_{\rho_2}}{K_{\rho_1}}\right)^2+\frac{3}{2}K_{c_{p1}}^2\right)\right]. \quad (24b)$$

The mathematical form of the solution is very similar to that for the Newtonian fluid case, see Merker and Mey [2]. This is not surprising as the solutions for the porous cavity at $O(\varepsilon^0)$ (see Blythe and Simpkins [3], and Bejan and Tien [5] was already very similar to that for a Newtonian fluid deduced by Cormack *et al.* [6].

In the solution given above the terms at $O(\varepsilon^1)$ and $O(\varepsilon^2)$ are corrections to the basic solution at $O(\varepsilon^0)$ which has been obtained by applying the Boussinesq approximation.

## 5. DISCUSSION

Figure 2 shows the relative deviation between the Nusselt number for variable properties and that for constant properties vs the expansion parameter $\varepsilon$ for $A = 0.02$ and the case $T_R = (T_h+T_c)/2$. It is interesting to note that the constant-property solution (Boussinesq approximation) becomes identical with the variable-property solution of the linear theory if the arithmetic mean between the cold and hot end wall temperature is used as the reference state. The

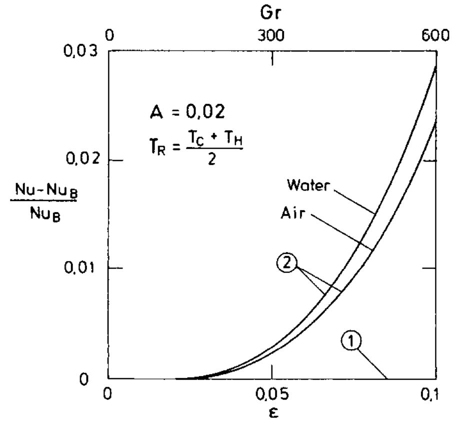


FIG. 2. Relative deviation between the variable-property and the constant-property solution vs the expansion parameter for $A = 0.02$ and in the case $T_R = (T_c+T_h)/2$: ①, linear theory; ②, quadratic theory.

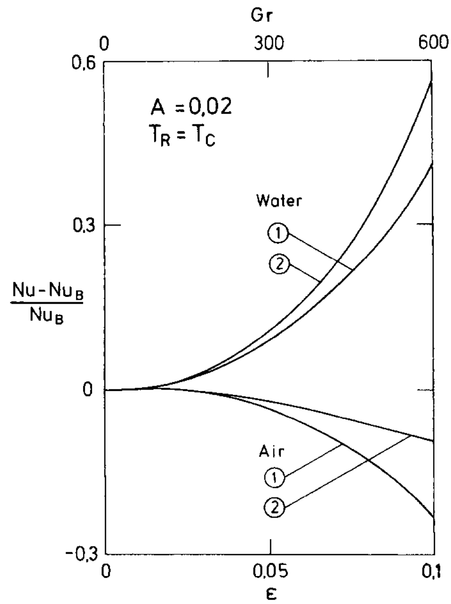


FIG. 3. Relative deviation between the variable-property and the constant-property solution vs the expansion parameter for $A = 0.02$ and in the case $T_R = T_c$: ①, linear theory; ②, quadratic theory.

Nusselt number is somewhat increased if the quadratic terms are added, that is approximately 2.5% for $\varepsilon = 0.10$, i.e. $Gr = 600$.

Figure 3 shows the results for the case $T_R = T_c$. The graphs marked ① indicate the results of the linear theory and those marked ② those of the quadratic theory. The graphs show that the effect of the variable properties on the Nusselt number is contrary for air and water. Considering the results of the linear theory for $\varepsilon = 0.1$, the Nusselt number for water is 42% larger and the Nusselt number for air is 24% lower than that obtained for constant properties. Adding the quadratic terms, the deviation for water becomes larger and that for air smaller. This rather strange behaviour shows clearly that in cavity flow problems the end wall temperature is not appropriate as a reference temperature.

Summing up, the results presented show that the arithmetic mean between the end wall temperature is a reasonable value as a reference temperature in cavity flow problems. The Nusselt numbers obtained from the constant-property solution (Boussinesq approximation) are sufficiently accurate if the variable properties are taken at this reference temperature.

## REFERENCES

1. H. Herwig, Asymptotische Theorie zur Erfassung des Einflusses variabler Stoffwerte auf Impuls- und Wärmeübertragung, *FortschrBer. VDI*, Reihe 7: *Strömungstechnik* Nr. 93 (1985).
2. G. P. Merker and St. Mey, Free convection in a shallow cavity with variable properties—1. Newtonian fluid, *Int. J. Heat Mass Transfer* **30**, 1825–1832 (1987).
3. P. A. Blythe and P. G. Simpkins, Convection in a porous layer for a temperature dependent viscosity, *Int. J. Heat Mass Transfer* **24**, 497–506 (1981).
4. K. L. Walker and G. M. Homsy, Convection in a porous cavity, *J. Fluid Mech.* **87**, 449–474 (1978).
5. A. Bejan and C. L. Tien, Natural convection in a horizontal porous medium subject to an end to end temperature difference, *J. Heat Transfer* **100**, 191–198 (1978).
6. D. E. Cormack, L. G. Leal and J. Imberger, Natural convection in a shallow cavity with differentially heated end walls. Part 1: asymptotic solution, *J. Fluid Mech.* **65**, 231–246 (1976).

## CONVECTION NATURELLE DANS UNE CAVITE PEU PROFONDE AVEC PROPRIETES VARIABLES—2. MILIEUX POREUX

**Résumé**—On étudie la convection naturelle dans une cavité poreuse peu profonde, avec des extrémités chauffées différemment. Les équations différentielles sont résolues analytiquement en appliquant la méthode des développements asymptotiques. Les résultats montrent que la solution pour propriétés constantes (approximation de Boussinesq) s'écarte approximativement de 3% de la solution à propriétés variables si les propriétés dans la solution de Boussinesq sont prises à la température moyenne entre celles des parois chaude et froide.

## FREIE KONVEKTION IN EINEM FLACHEN BEHÄLTER MIT VARIABLEN STOFFWERTEN—2. PORÖSE MEDIEN

**Zusammenfassung**—Die freie Konvektion in einem porösen und flachen Behälter, dessen Stirnflächen unterschiedlich beheizt sind, wird theoretisch untersucht. Die das Problem beschreibenden Differentialgleichungen werden mit der Methode der angepaßten asymptotischen Entwicklung analytisch gelöst. Die Ergebnisse zeigen, daß die Lösung für konstante Stoffwerte (Boussinesq-Näherung) um maximal 3% von der für veränderliche Stoffwerte abweicht, wenn die Stoffwerte in die Boussinesq-Lösung bei der arithmetischen Mitteltemperatur eingesetzt werden.

## ЕСТЕСТВЕННАЯ КОНВЕКЦИЯ В МЕЛКОЙ ПОЛОСТИ С УЧЕТОМ ПЕРЕМЕННОСТИ СВОЙСТВ—2. ПОРИСТАЯ СРЕДА

**Аннотация**—Проведено исследование естественной конвекции в пористой мелкой полости, торцевые стенки которой поддерживаются при разной температуре. С помощью метода асимптотических разложений решение для зависящих от температуры свойств (приближение Буссинеска) отличается, примерно, на 3% от решения для случая переменных свойств, если эти свойства в решении Буссинеска берутся при средне-арифметическом значении температуры между горячей и холодной торцевыми стенками.